Electromagn... oscillations from a bent antenna ...

Robert Cameron Colwell

ELECTROMAGNETIC OSCILLATIONS FROM A BENT ANTENNA

A DISSERTATION

PRESENTED TO THE

FACULTY OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

BY

ROBERT CAMERON COLWELL

PRESS OF
THE NEW ERA PRINTING COMPANY
LANCASTER, PA.

1920

Accepted by the Department of
Physics, February, 1918

ELECTROMAGNETIC OSCILLATIONS FROM A BENT ANTENNA

The purpose of this investigation is to find the mathematical equations for the electromagnetic oscillations from a bent antenna, which is known to send out directed waves. The method used is that of Pocklington[1] which has been developed by M. Abraham[2] and particularly by G. W. Peirce[3] who has recently published a remarkable research on the radiation resistance of a flat top antenna. This article is based upon the work of Peirce and Abraham, but the equations are worked out for the fundamental and not for the forced vibration. The application of the formula is new.

The following assumptions are made:

1. That any antenna may be considered to be made up of a large number of Hertzian doublets placed end to end.

2. That the earth is a perfectly conducting plane.

3. That the waves propagated high into the air eventually return to the earth. The reason for this assumption will be shown in the section dealing with the horizontal part of the antenna.

Let a flat top antenna have a vertical part h and a horizontal part d, Fig. 1. It will be necessary to discuss the effect of the

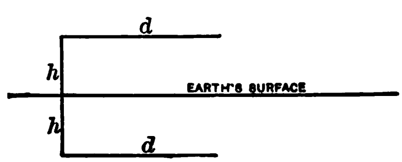


Fig. 1

radiation from this antenna in three parts.

I. The radiation due to the vertical part h and its image $-h$.

[1] Pocklington H. C., Camb. Phil. Soc. Proc., 1898, p. 325.

[2] Abraham, Theorie der Electrizitat, Vol. II.

[3] G. W. Peirce, "Radiation Characteristics of an Antenna," Proc. Am. Acad. Arts and Sciences, Vol. 52, No. 4, October, 1916.

II. The radiation from the horizontal part d and its image $-d$.
III. The mutual action between the vertical and horizontal parts.

I. Radiation from the Vertical Part

Let the point where the vertical part enters the ground be the origin of co-ordinates, and choose the axes as shown in Fig. 2

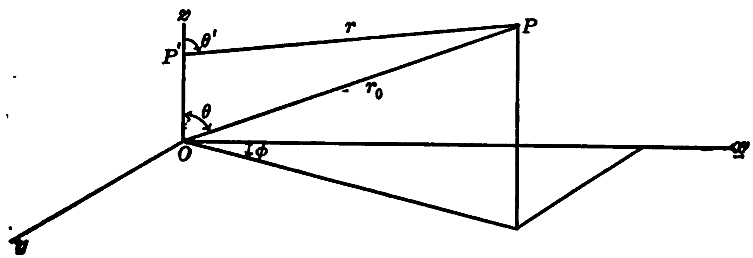


Fig. 2

where the x axis is parallel to the direction in which the free end of the horizontal antenna points and the vertical antenna coincides with the direction OZ. The azimuth and co-latitude angles have their usual designations. Let P be any point in space whose polar co-ordinates are r, ϕ, θ, where r is very great compared to the height h of the vertical part of the antenna. P lies at a distance r from the point P' which is the position of one of the Hertzian doublets postulated in the first assumption. The effect of the doublet at P' on the point P is given by the theory of Hertz[1]

$$dE_\theta = dH_\phi = \frac{\sin\theta}{V^2 r_0} f''\left(t - \frac{r}{V}\right) \qquad (1)$$

where $f(t)$ = the moment of the doublet edz, r = length PP' and V = velocity of light. E is expressed in electrostatic and H in electromagnetic units; r_0 and θ appear in place of r and θ', as is legitimate because of the great magnitude of r in comparison with OP.

If the total length of the antenna is $l = h + d$ there must be a node of current at $+l$ and $-l$. For the fundamental vibra-

[1] Hertz, Electric waves, Chap. IX, Trans. D. E. Jones; Bateman, Electrical and Optical Wave Motion, p. 8.

tion, the current i at any time expressed in terms of I the maximum current must be

$$i = \mathrm{I} \sin pt \cos \frac{2\pi z}{\lambda} \tag{2}$$

where $\lambda =$ the wave length of the system,

$$p = \frac{2\pi V}{\lambda} \text{ is the angular velocity}$$

Now

$$\frac{d^2}{dt^2}(ft) = \frac{d^2e}{dt^2}dz$$

and the current i is the rate of change of the charge e on any doublet, that is

$$i = \frac{de}{dt}, \qquad \frac{di}{dt} = \frac{d^2e}{dt^2}$$

Therefore

$$\frac{d^2e}{dt^2} = \frac{di}{dt} = \frac{2\pi VI}{\lambda} \cos \frac{2\pi Vt}{\lambda} \cos \frac{2\pi z}{\lambda} \tag{3}$$

Substituting (3) in (1) we get

$$dE_\theta{}^a = \frac{pI \sin \theta}{V^2 r_0} \cos p \left(t - \frac{r_0 - z \cos \theta}{V} \right) \cos \frac{2\pi z}{\lambda} dz \tag{4}$$

in which $\frac{2\pi V}{\lambda}$ is given the shorter form p, and $r_0 - z \cos \theta$ is written for r as these are approximately equal.

The doublet[1] at OP' has an image at $- z$ and this doublet will have an effect at P given by the equation

$$dE_\theta{}^i = \frac{pI \sin \theta}{V^2 r_0} \cos p \left(t - \frac{r + z \cos \theta}{V} \right) \cos \frac{2\pi z}{\lambda} dz \tag{5}$$

The effect due to the two doublets on the point P is found by adding (4) and (5), and the action of all the doublets and their

[1] Theory of Images: Jeans, Electricity and Magnetism, Chapter VIII; Maxwell, Electricity and Magnetism, Chapter XI; Webster, Electricity and Magnetism, p. 303.

images, by integrating from 0 to h. Then

$$E_\theta = H_\phi = \frac{pI \sin\theta}{V^2 r_0} \int_0^h \cos\frac{2\pi z}{\lambda}\left[\cos\left\{p\left(t - \frac{r_0}{V}\right) + \frac{pz\cos\theta}{V}\right\} - \cos\left\{p\left(t - \frac{r_0}{V}\right) - \frac{pz\cos\theta}{V}\right\}\right]dz$$

The terms in the square brackets are of the form [Cos $(x + y)$ $-$ cos $(x - y)$] may be simplified by use of a well-known trigonometrical formula: then

$$E_\theta = \frac{2pI\sin\theta}{V^2 r_0}\cos p\left(t - \frac{r_0}{V}\right)\int_0^h \cos\frac{pz\cos\theta}{V}\cos\frac{2\pi z}{\lambda}\,dz \quad (7)$$

The integral in (7) is a standard form and is found in any table of integrals. Integrating and putting in the limits we get for the integral of (7)

$$\frac{\sin\left(\frac{ph\cos\theta}{V} - \frac{2\pi h}{\lambda}\right)}{2\left(\frac{p\cos\theta}{V} - \frac{2\pi}{\lambda}\right)} + \frac{\sin\left(ph\cos\theta + \frac{2\pi h}{\lambda}\right)}{2\left(\frac{p\cos\theta}{V} + \frac{2\pi}{\lambda}\right)}$$

This reduces to

$$\frac{\sin\left[\frac{2\pi h}{\lambda}(\cos\theta - 1)\right]}{\frac{4\pi}{\lambda}(\cos\theta - 1)} + \frac{\sin\left[\frac{2\pi h}{\lambda}(\cos\theta + 1)\right]}{\frac{4\pi}{\lambda}(\cos\theta + 1)}$$

Adding these fractions we get for the integrated term

$$\frac{\lambda}{2\pi\sin^2\theta}\left\{\cos\left(\frac{2\pi h}{\lambda}\cos\theta\right)\sin\frac{2\pi h}{\lambda} - \sin\left(\frac{2\pi h}{\lambda}\cos\theta\right)\cos\frac{2\pi h}{\lambda}\cos\theta\right\}$$

Substituting in (7)

$$E_\theta = \frac{2I}{Vr_0\sin\theta}\cos\frac{2\pi}{\lambda}(Vt - r_0)\left\{\cos\left(\frac{2\pi h}{\lambda}\cos\theta\right)\sin\frac{2\pi h}{\lambda} - \cos\theta\sin\left(\frac{2\pi h\cos\theta}{\lambda}\right)\cos\frac{2\pi h}{\lambda}\right\} \quad (8)$$

Equation (8) is the basis for the following brief discussion of a vertical antenna. Poynting's Vector Theorem states that the power radiated along the radius at any point of a sphere is $V/4\pi(E \times H)$ per unit area. In the case of the vertical $E = H$. The energy radiated is directly proportional to E_θ. The wave length is equal to $4h$. $2I/Vr_0$ is constant. The average value of $\cos^2 2\pi/\lambda(Vt - r_0)$ is $\frac{1}{2}$. Therefore the expression for the power radiated at an angle θ takes the simple form

$$P_\theta = K\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]^2 \tag{9}$$

The angle at which the greatest amount of energy is radiated is given by

$$\frac{dP_\theta}{d\theta} = 2K\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\left[\frac{\sin\left(\frac{\pi}{2}\cos\theta\right)\frac{\pi}{2}\sin^2\theta - \cos\left(\frac{\pi}{2}\cos\theta\right)\cos\theta}{\sin^2\theta}\right] = 0 \tag{10}$$

The solution $\theta = \pi/2$ makes $d^2P_\theta/d\theta^2$ negative, and therefore represents the direction of maximum radiation of energy.

Therefore a vertical antenna sends out its maximum energy along the horizontal. Table 1 is developed from (9), K being given an arbitrary value. Diagram A is a graph of Table I, showing the approximate energy distribution. The distribution of energy is such that nearly all the radiation lies within the 30° angle with the horizon. These equations (8) and (9) throw some light on the action of tall antennas. The radius of action is increased as the antenna height is increased for two reasons. First, because the lengthening of the antenna automatically increased the wave length and the earth is a better conductor for long waves than for short ones. Secondly, because the waves from a high antenna are not brought to earth so quickly (see assumption three) as those from a short one. Just as soon, however, as the wave length is reached for which the earth is a perfect conductor, no further improvement in conduction is effected by increasing the wave. Similarly there is a limit to the second action, so that there is no reason for increasing the height of the sending antenna indefinitely.

TABLE I

Angle	E_θ^2	E_θ
0	0.00	0.00
15	.078	.22
30	.144	.42
45	.422	.65
60	.672	.82
75	.902	.95
90	1.00	1.00

II. THE HORIZONTAL PART

In order to apply Hertz's theory to the horizontal antenna, it is necessary to measure the co-latitude from the axis X. The co-latitude is designated by ψ and the azimuth by χ. The origin remains at the point where the vertical antenna enters the earth, Fig. 3. It should be noticed here that the action of

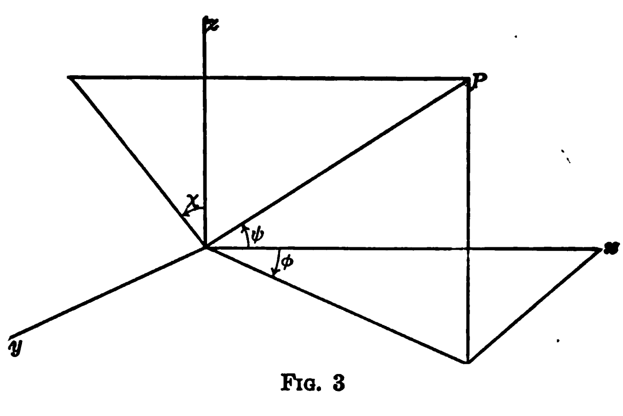


FIG. 3

the horizontal part is dependent upon the length of the vertical for the latter determines the phase of the current at any point in the former. Neither in theory nor in practice can the action of the horizontal be dissociated from the vertical. The positions of the doublets and doublet images are denoted in Fig. 4. The small letters x, y, z (or x, o, h) refer to the doublets, while the position of the point in space P is given by large X, Y, Z.

Hertz's theory may now be applied to the effect at P of the doublets at Q and Q'. Then

$$dE_\psi = dH_x = \frac{\sin\psi}{r_0 V^2}\left\{f''\left(t - \frac{r_1}{V}\right) + F''\left(t - \frac{r_2}{V}\right)\right\} \quad (10)$$

Now

$$r_1 = \sqrt{(X - x)^2 + Y^2 + (Z - h)^2} \quad (11)$$

$$r_2 = \sqrt{(X - x)^2 + Y^2 + (Z + h)^2} \quad (12)$$

At great distances

$$2r_0 = r_1 + r_2 \quad (13)$$

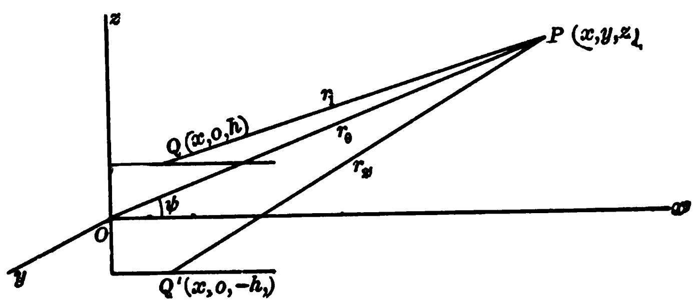


FIG. 4

From (11), (12), (13)

$$r_1 = r_0 + \frac{x^2 - 2Xx - 2Zh + h^2}{2r} \quad (14)$$

$$r_2 = r_0 + \frac{x^2 - 2Xx + 2Zh + h^2}{2r} \quad (15)$$

As before in (3)

$$f''\left(t - \frac{r_1}{V}\right) = p\,\mathrm{I}\cos p\left(t - \frac{r_1}{V}\right)\cos\frac{2\pi}{\lambda}(h - x)dx$$

$$= \frac{2\pi V}{\lambda} I \cos\frac{2\pi}{\lambda}\left(Vt - r_0 - \frac{x^2 - 2Xx - 2Zh + h^2}{2r_0}\right)\cos\frac{2\pi}{\lambda}(h + x)dx \quad (16)$$

Also,

$$F''\left(t - \frac{r_2}{V}\right) = -\,p\,\mathrm{I}\cos p\left(t - \frac{r_2}{V}\right)\cos\frac{2\pi}{\lambda}(h + x)dx$$

$$= -\frac{2\pi VI}{\lambda}\cos\frac{2\pi}{\lambda}\left\{Vt - r_0\right.$$

$$-\frac{(x^2 - 2Xx + 2Zh + h^2)}{2r_0}\Bigg\} \cos\frac{2\pi}{\lambda}(h+x)dx \quad (17)$$

Substituting (16) and (17) in (10), simplifying by the trigonometrical formula for the sum of two cosines; neglecting $x^2 + h^2$ in comparison with $2Xx$, we obtain the expression

$$dE_\psi = dH_\chi = -\frac{4\pi I \sin\psi}{r_0 V\lambda}\cdot\sin\frac{2\pi Zh}{\lambda r_0}$$

$$\cdot\left[\sin\frac{2\pi}{\lambda}\{Vt - r_0 + x\cos\psi\}\cos\frac{2\pi}{\lambda}(h+x)dx\right] \quad (18)$$

The integration of (18) is very similar to that of (7) and gives finally

$$E_\psi = \frac{2I}{r_0 V\sin\psi}\sin\frac{2\pi Zh}{\lambda r_0}\left[\sin\frac{2\pi}{\lambda}(Vt - r_0)\left\{\cos\frac{2\pi d}{\lambda}\right.\right.$$

$$\left.-\cos\left(\frac{2\pi d}{\lambda}\cos\psi\right)\right\} + \cos\frac{2\pi}{\lambda}(Vt - r_0)$$

$$\left.\left.\times\left\{\cos\psi\sin\frac{2\pi d}{\lambda} - \sin\left(\frac{2\pi d}{\lambda}\cos\psi\right)\right]\right\} \quad (19)$$

Upon equations (8) and (19) are based all the conclusions regarding a bent antenna. The total flux of force is obtained by compounding E_θ of (8) with the E_ψ of (19) in such a way as to get the complete electric intensity E, and the corresponding magnetic intensity H. Although the doublets which give rise to E_θ are perpendicular to those producing E_ψ, it by no means follows that E_ψ and E_θ are at right angles; in fact, they make an angle (α) with one another which varies for every point on the particular sphere under consideration. Professor Peirce, of Harvard University, has shown that this difference in direction gives rise to a third term in the power radiation {the other two come of course from (8) and (19)} of the form

$$2\cos\alpha E_\theta E_\psi, \quad \text{where} \quad \cos\alpha = -\frac{\sin\psi\cos\theta\cos\phi}{1 - \sin^2\theta\cos^2\phi}$$

The average value of this term, when multiplied by $V/4\pi$, he calls the *mutual power*. In the problem under discussion

$$\frac{2V\cos\alpha}{4\pi}E_\theta E_\psi = \frac{-2K\cos\theta\cos\phi}{\sin\theta\,(1-\sin^2\theta\cos\phi)}\sin\frac{2\pi hZ}{\lambda r_0}\left\{\cos\psi\sin\frac{2\pi d}{\lambda} - \sin\left(\frac{2\pi d}{\lambda}\cos\psi\right)\right\}$$
$$\times\left\{\sin\frac{2\pi h}{\lambda}\cos\frac{2\pi h\cos\theta}{\lambda} - \cos\theta\cos\frac{2\pi h}{\lambda}\sin\frac{\pi 2h\cos\theta}{\lambda}\right\} \quad (20)$$

where for convenience K is put equal to $I^2/\pi V r_0^2$. This equation is true only for the *average* value of $(V/2\pi)\cos\alpha E_\theta E_\psi$ so that the term

$$\cos^2\frac{2\pi}{\lambda}(Vt-r_0) = \tfrac{1}{2} \text{ and } \sin\frac{2\pi}{\lambda}(Vt-r_0)\cos\frac{2\pi}{\lambda}(Vt-r_0) = 0$$

SECOND PART—NUMERICAL CALCULATION

The power radiated per unit area in certain zones at distances 2,000, 6,000 and 9,000 meters above the earth's surface, and at a distance 10,000 meters from the origin will now be calculated by means of equations (8), (19) and (20). In these equations let $h = 10$ meters, $d = 100$ meters, $\lambda = 4(h+d) = 440$. Then the power radiated from the vertical part may be expressed by (8) in the form:

$$\frac{V\,E_\theta^2}{4\pi} = K\left\{\cos\left(\frac{2\pi h}{\lambda}\cos\theta\right)\sin\frac{2\pi h}{\lambda} - \cos\theta\sin\left(\frac{2\pi h}{\lambda}\cos\theta\right)\cos\frac{2\pi h}{\lambda}\right\}^2$$

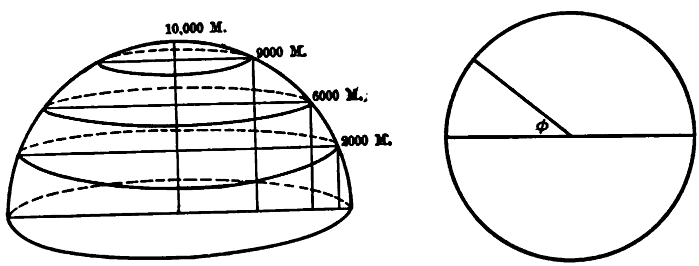


FIG. 5

The values in Table II are calculated from this equation:

VERTICAL—TABLE II

	Height	$E_\phi^2 \frac{V}{4\pi}$
$\lambda =$ 440M	9000M	.0049K
$h =$ 10M	6000M	.0160
$r =$ 10000M		
	5000	.0164
	2000	.0196
	0	.0196

The calculations of $E_\phi^2(V/4\pi)$ for the horizontal part are by no means as simple as this. The value of the angle in formula (19) changes for every point on the zone at the height 9000, 6000 and so on. This change must be calculated. In Fig. 6 ABC

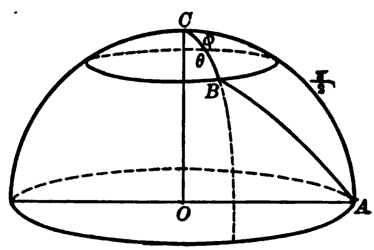


FIG. 6

is a spherical triangle with sides θ, ψ, $\pi/2$ and angle ϕ given. Then from a well-known formula in the trigonometry of triangles:

$$\text{Cos}\,\psi = \sin\theta\cos\phi$$

The power radiated from the horizontal antenna through the zone $Z = 9000$, 6000, etc., will vary for different angles of ϕ, that is, the bent part of the antenna has a directive effect on the oscillations.

In equation (19)

$$\frac{VE^2}{4\pi} = K\frac{\sin^2\left\{\dfrac{2\pi Zh}{\lambda r_0}\right\}}{\sin^2\psi}\left\{\cos^2\frac{2\pi d}{\lambda} + 1 + \cos^2\psi\sin^2\frac{2\pi d}{\lambda}\right.$$
$$\left. - 2\cos\frac{2\pi d}{\lambda}\cos\left(\frac{2\pi d}{\lambda}\cos\right) - 2\cos\psi\sin\frac{2\pi d}{\lambda}\sin\frac{2\pi d}{\lambda}\cos\psi\right\}$$

From this equation the values of Table III are calculated for the different zones and different directions. The operations are long and tedious but not difficult. The horizontal part of the antenna has a slight directive effect perpendicular to the direction in which it points. This is contrary to what one would expect: because all experiments show a directive effect in the direction away from the free end of the antenna. However, it has been shown by Pocklington that a circular wire radiates more power perpendicular to its area than in any other direction and the form of equation (19) shows that Pocklington's method of doublets applies to this problem and that the solution is correct to the approximations made.

HORIZONTAL PART—TABLE III

Height	ϕ	$E_\psi^2 \frac{V}{4\pi}$	ϕ
9000	0	.0097	180
	30	.0094	150
	45	.0092	135
	60	.0097	120
	90	.0107	90
6000	0	.0022	180
	30	.0028	150
	45	.0031	135
	60	.0040	120
	90	.0047	90
2000	0	.0004	180
	30	.0001	150
	60	.0003	120
	90	.0000	90

It will now be shown that the power arising from the mutual effect (eq. 20) tends to modify the directive effect at high points in the atmosphere in such a way as to give a fore and aft directive effect.

The values of Table IV are calculated from equation (20).

MUTUAL EFFECT—TABLE IV

Height.	ϕ	$\frac{VE_\psi E_\phi}{4\eta}$	ϕ
9000	0	.0027K	180
	30	.0022	150
	45	.0014	135
	60	.0005	120
	90	——	90
6000	0	.0030	180
	30	.0023	150
	45	.0015	135
	60	.0009	120
	90	——	90
2000		Negligible	

Adding up the vertical, the horizontal and the mutual effects contained in Tables II, III and IV, we obtain the complete power radiated through unit area of the zones. The results are given in Table V, in which *K* has been set equal to an arbitrary value. Plate *B* is plotted from Table V. In Plate *C*, the distribution at 9000 *M* is compared to a curve obtained experimentally by Fleming.

TOTAL EFFECT—TABLE V

Height	ϕ	P	ϕ
9000	0	.0173	180
	30	.0165	150
	45	.0155	135
	60	.0151	120
	90	.0156	90
6000	0	.0212	180
	30	.0211	150
	45	.0206	135
	60	.0209	120
	90	.0207	90
2000	0	.0206	180
	30	.0197	150
	60	.0199	120
	90	.0196	90

The symmetry of the curves developed from the theory can be reconciled with the asymmetrical curve found in the experiments by supposing that the electrical waves brought back to the earth from high in the atmosphere are more intense toward the bend in the antenna than at the free end. Zenneck (Zenneck, Phys. Zeitsch., Vol. 9, p. 553, 1908) has shown that this difference may be due to imperfect conductivity in the earth's surface.

SUMMARY

First: The equations developed by Pocklington, Abraham and Peirce have been applied to an antenna with vertical and horizontal parts in such a way as to find the energy given out for the fundamental vibration:

$$\lambda = 4(h + d)$$

Second: The intensities so obtained are plotted and are shown to have a fore and aft directive inclination. The intensity fore and aft is symmetrical and not asymmetrical as required by the experiments.

Third: Close to the antenna the intensities are symmetrical in azimuth agreeing with experiment.

Fourth: The conclusion is that the fore and aft asymmetry of a bent antenna is caused by the difference in conductivity between the atmosphere and the earth for the electromagnetic oscillation.

Fifth: The forms of the resulting equations show that Peirce's assumptions regarding the current satisfy Pocklington's criterion for the use of Maxwell's Equations.

Addendum

The Integration of Equations (18), *p.* 10.

The integrable part of (18) is

$$\int \sin\frac{2\pi}{\lambda}\{Vt - r_0 + x \operatorname{Cos}\psi\} \cos\frac{2\pi}{\lambda}(h + x)dx$$

Expand and multiply, thus:

$$\int_0^d \left[\sin\frac{2\pi}{\lambda}(Vt-r_0)\cos\frac{2\pi x \operatorname{Cos}\psi}{\lambda} + \cos\frac{2\pi}{\lambda}(Vt-r_0)\sin\frac{2\pi x \operatorname{Cos}\psi}{\lambda}\right]$$

$$\cdot\left[\cos\frac{2\pi h}{\lambda}\cos\frac{2\pi x}{\lambda} - \sin\frac{2\pi h}{\lambda}\frac{\sin 2\pi x}{\lambda}\right]dx$$

$$= \int_0^d \sin\frac{2\pi}{\lambda}(Vt - r_0)\cos\frac{2\pi h}{\lambda}\cos\frac{2\pi x \operatorname{Cos}\psi}{\lambda}\cos\frac{2\pi x}{\lambda}dx \quad (21)$$

$$+ \cos\frac{2\pi}{\lambda}(Vt - r_0)\cos\frac{2\pi h}{\lambda}\int_0^d \sin\frac{2\pi x \operatorname{Cos}\psi}{\lambda}\cos\frac{2\pi x}{\lambda}dx \quad (22)$$

$$- \sin\frac{2\pi}{\lambda}(Vt - r_0)\sin\frac{2\pi h}{\lambda}\int_0^d \cos\frac{2\pi x \operatorname{Cos}\psi}{\lambda}\sin\frac{2\pi x}{\lambda}dx \quad (23)$$

$$- \cos\frac{2\pi}{\lambda}(Vt - r_0)\sin\frac{2\pi h}{\lambda}\int_0^d \sin\frac{2\pi x \operatorname{Cos}\psi}{\lambda}\sin\frac{2\pi x}{\lambda}dx \quad (24)$$

The integration of (21) is the same as that of (7) on page 4 and gives

$$\frac{\lambda}{2\pi \sin^2\psi}\left[\cos\frac{2\pi d \cos\psi}{\lambda}\sin\frac{2\pi d}{\lambda} - \cos\psi\sin\frac{2\pi d\cos\psi}{\lambda}\cos\frac{2\pi d}{\lambda}\right] \quad (25)$$

The integration of the integral part (22) comes under Form 360 in Peirce's Table of Integral and simplifies into

$$\frac{\lambda}{2\pi \sin^2\psi}\left\{\cos\psi \cos\frac{2\pi d \cos\psi}{\lambda}\cos\frac{2\pi d}{\lambda} + \sin\frac{2\pi d\cos\psi}{\lambda}\sin\frac{2\pi h}{\lambda} - \cos\psi\right\} \quad (26)$$

(23) comes under form (360) but in a different way from (22). The complete integration follows:

To find

$$\int_0^d \sin\frac{2\pi x}{\lambda}\cos\frac{2\pi x \operatorname{Cos}\psi}{\lambda}\,dx$$

(Form 360)

$$\int \sin mx \cos nx\, dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}$$

Let

$$m = \frac{2\pi}{\lambda}, \quad n = \frac{2\pi\cos\psi}{\lambda}$$

Then integration gives:

$$\left[-\frac{\cos\left(\frac{2\pi}{\lambda}-\frac{2\pi}{\lambda}\cos\psi\right)x}{2\left\{\frac{2\pi}{\lambda}-\frac{2\pi}{\lambda}\cos\psi\right\}} - \frac{\cos\left(\frac{2\pi}{\lambda}+\frac{2\pi\cos\psi}{\lambda}\right)x}{2\left\{\frac{2\pi}{\lambda}+\frac{2\pi}{\lambda}\cos\psi\right\}}\right]_0^d$$

$$= \frac{-\lambda}{4\pi}\left\{\frac{\cos\left(\frac{2\pi}{\lambda}-\frac{2\pi\cos\psi}{\lambda}\right)d}{1-\cos\psi} + \frac{\cos\left(\frac{2\pi}{\lambda}+\frac{2\pi}{\lambda}\cos\psi\right)}{1+\cos\psi}\right\} + \frac{\lambda}{2\pi\sin^2\psi}$$

$$= \frac{\lambda}{4\pi}\left\{\begin{array}{l} (1+\cos\psi)\left\{\cos\frac{2\pi d}{\lambda}\cos\frac{2\pi d \operatorname{Cos}\psi}{\lambda} + \sin\frac{2\pi d}{\lambda}\sin\frac{2\pi d\cos\psi}{\lambda}\right\} \\ (1-\cos\psi)\left\{\cos\frac{2\pi d}{\lambda}\cos\frac{2\pi d\cos\psi}{\lambda} - \sin\frac{2\pi d}{\lambda}\sin\frac{2\pi d\cos\psi}{\lambda}\right\}\end{array}\right\} + \frac{\lambda}{2\pi\sin^2\psi}$$

$$= \frac{-\lambda}{2\pi \sin^2 \psi} \left\{ \begin{aligned} &\cos \frac{2\pi d}{\lambda} \cos \frac{2\pi d \cos \psi}{\lambda} \\ &\qquad + \cos \psi \sin \frac{2\pi d}{\lambda} \sin \frac{2\pi d \cos \psi}{\lambda} - 1 \end{aligned} \right\} \tag{27}$$

The integrable part of (24)

$$\int_0^d \sin \frac{2\pi x \operatorname{Cos} \psi}{\lambda} \sin \frac{2\pi x}{\lambda} dx$$

is also a standard form and becomes finally

$$\frac{\lambda}{2\pi \sin^2 \psi} \left\{ \cos \psi \cos \frac{2\pi d \cos \psi}{\lambda} \sin \frac{2\pi d}{\lambda} - \sin \frac{2\pi d \cos \psi}{\lambda} \cos \frac{2\pi d}{\lambda} \right\} \tag{28}$$

Substitute 25, 26, 27 and (28) in (21), (22), (23) and (24). Then insert the result in (18) and obtain:

$$E_\psi = \frac{-4\pi I \sin \psi}{r_0 V \lambda} \sin \frac{2\pi Z h}{\lambda r_0} \cdot \frac{\lambda}{2\pi \sin^2 \psi}$$

$$\begin{aligned} \Bigg[&\sin \frac{2\pi}{\lambda} (Vt - r_0) \cos \frac{2\pi h}{\lambda} \left\{ \cos \frac{2\pi d \cos \psi}{\lambda} \sin \frac{2\pi d}{\lambda} \right. \\ &\qquad \left. - \cos \psi \sin \frac{2\pi d \operatorname{Cos} \psi}{\lambda} \cos \frac{2\pi d}{\lambda} \right\} \\ &+ \cos \frac{2\pi}{\lambda} (Vt - r_0) \cos \frac{2\pi h}{\lambda} \left\{ \cos \frac{2\pi d \cos \psi}{\lambda} \cos \frac{2\pi d}{\lambda} \right. \\ &\qquad \left. + \sin \frac{2\pi d \cos \psi}{\lambda} \sin \frac{2\pi h}{\lambda} - \cos \psi \right\} \\ &+ \sin \frac{2\pi}{\lambda} (Vt - r_0) \sin \frac{2\pi h}{\lambda} \left\{ \cos \frac{2\pi d}{\lambda} \cos \frac{2\pi d \cos \psi}{\lambda} \right. \\ &\qquad \left. + \cos \psi \sin \frac{2\pi d}{\lambda} \sin \frac{2\pi d \cos \psi}{\lambda} - 1 \right\} \\ &+ \cos \frac{2\pi}{\lambda} (Vt - r_0) \sin \frac{2\pi h}{\lambda} \left\{ - \cos \psi \frac{\cos 2\pi d \cos \psi}{\lambda} \sin \frac{2\pi d}{\lambda} \right. \\ &\qquad \left. + \sin \frac{2\pi d \cos \psi}{\lambda} \cos \frac{2\pi d}{\lambda} \right\} \Bigg] \end{aligned} \tag{29}$$

Now

$$h + d = \frac{\lambda}{4} \therefore \frac{2\pi h}{\lambda} = \frac{\pi}{2} - \frac{2\pi d}{\lambda}$$

so that

$$\cos \frac{2\pi h}{\lambda} = \sin \frac{2\pi d}{\lambda}$$

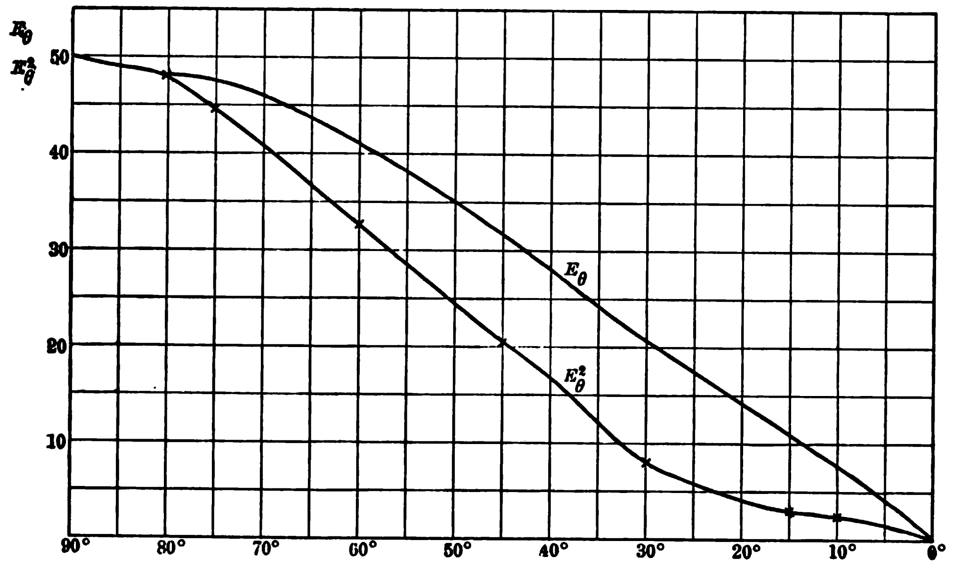
$$\sin \frac{2\pi h}{\lambda} = \cos \frac{2\pi d}{\lambda}$$

When these expressions are substituted in (29), the terms in $\sin (2\pi/\lambda)(Vt - r_0)$ and those in $\cos (2\pi/\lambda)(Vt - r_0)$ may be added together and give the simple form:

$$E_\psi = -\frac{2I}{r_0 V \sin \psi} \sin \frac{2\pi Z h}{\lambda r_0} \left[\sin \frac{2\pi}{\lambda}(Vt - r_0) \left\{ \cos \frac{2\pi d \cos \psi}{\lambda} - \cos \frac{2\pi d}{\lambda} \right\} + \cos \frac{2\pi}{\lambda}(Vt - r_0) \left\{ - \cos \psi \sin \frac{2\pi d}{\lambda} + \sin \frac{2\pi d \cos \psi}{\lambda} \right\} \right]$$

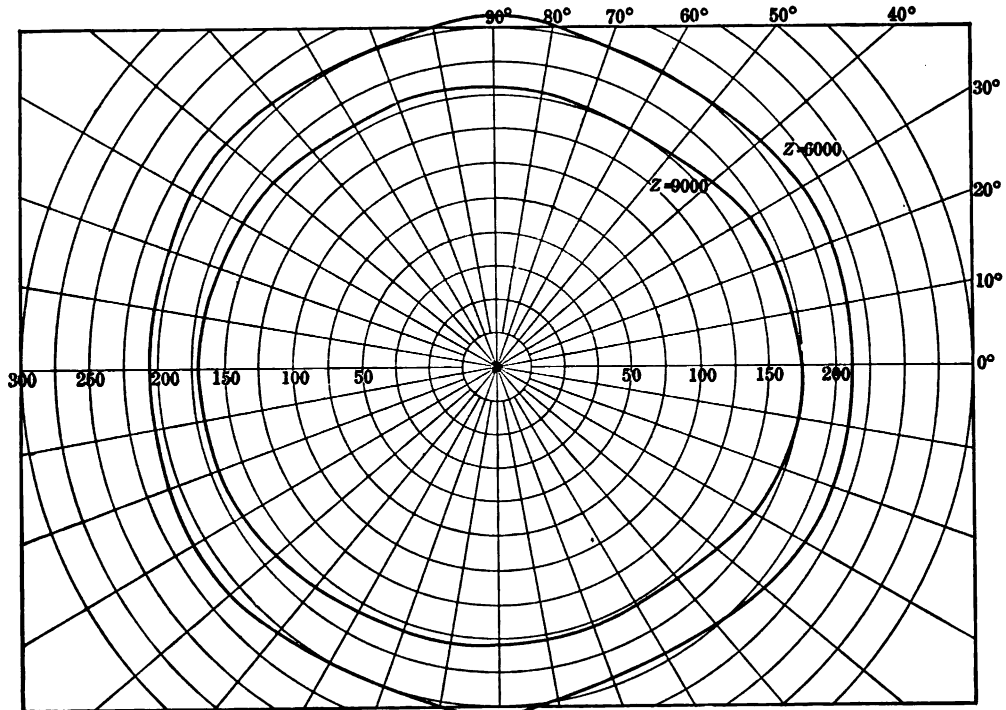
which is equation (19), page (9).

PLATE A



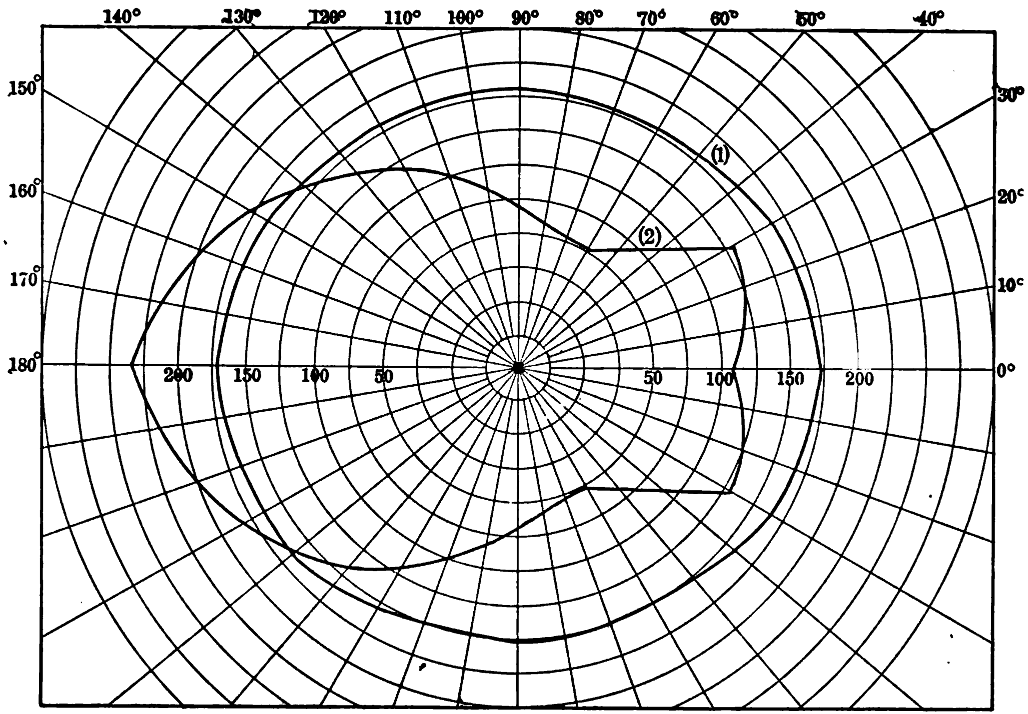
Variation of $E\theta$ with θ for fundamental

Plate B



90°
80°
70°
60°
50°
40°
30°
20°
10°
0°
Z=6000
Z=9000
300
250
200
150
100
50
50
100
150
200

Plate C



(1) Theoretical Curve. (2) Experimental Curve